Chapter 10: Counting

Cardinality of finite sets

When we 'count' the elements of a finite set, we are effectively constructing a bijection from the set

$$\mathbb{N}_n := \{1, 2, \dots, n\} = \{i \in \mathbb{Z} \mid 1 \leq i \leq n\}$$

to the set being counted.

Definition

Given a set X, if there is a bijection $f: \mathbb{N}_n \to X$ then we say that the cardinality of X (the number of elements in X) is n. We write

$$|X| = n.$$

Example: Let $X = \{7, 8, 9, 10, 11\}$.

Then $|X| = 5$ because $\exists f: \mathbb{N}_5 \to X$, where f is a bijection. In particular, we may choose
$f(1) = 7$, $f(2) = 8$, $f(3) = 9$, $f(4) = 10$, $f(5) = 11$.

Of course, there are many other bijections between $\mathbb{N}_5$ and X. Are there bijections between $\mathbb{N}_m$ and X with $m \neq n$? If there are, we're in trouble bec. a set would have two cardinalities.

Proposition: Suppose $f : \mathbb{N}_m \to X$ and $g : \mathbb{N}_n \to X$ are both bijections. Then $m = n$.

Strategy: Since g is a bijection, g^{-1} is a bijection ($g^{-1} : X \to \mathbb{N}_n$).

So $g^{-1} \circ f : \mathbb{N}_m \to \mathbb{N}_n$ is a bijection. (why?)

So we would be done if we can prove that (having a bijection $h : \mathbb{N}_m \to \mathbb{N}_n$) $\Rightarrow$ ($m = n$). —— (*)

We can prove (*) by showing:

($h : \mathbb{N}_m \to \mathbb{N}_n$ is an injection) $\Rightarrow$ ($m \leq n$). —— (**)

In particular, to prove (*) using (**) we observe that:

h is a bijection $\Rightarrow$ h is an injection. $\Rightarrow m \leq n$

h is a bijection $\Rightarrow h^{-1}$ is a bijection $\Rightarrow h^{-1}$ is an injection with $h^{-1}: \mathbb{N}^n \to \mathbb{N}^m$

$\Rightarrow n \leq m$

We would then have $(n \leq m) \wedge (m \leq n) \Rightarrow (m = n)$ so we'd be done. So now the whole proof would work provided (**) holds.

Lemma: If there is an injection $\mathbb{N}_m \to \mathbb{N}_n$ then $m \leq n$

{the Pigeonhole principle}

We will prove this lemma shortly. For now, let us assume it is true, and prove the proposition.

Proof of proposition: Since $f: \mathbb{N}_m \to X$ and $g: \mathbb{N}_n \to X$ are bijections, they are invertible and their inverses $f^{-1}: X \to \mathbb{N}_m$ & $g^{-1}: X \to \mathbb{N}_n$ are also bijections.

Thus $g^{-1} \circ f: \mathbb{N}_m \to \mathbb{N}_n$ is a bijection, hence an injection. By the lemma above, we now have $m \leq n$.

Similarly $f^{-1} \circ g: \mathbb{N}_n \to \mathbb{N}_m$ is a bijection, hence an injection. By the lemma above, we now have $n \leq m$.

Together, $n \leq m$ & $m \leq n$ yield $m = n$.

Now, let us go back and prove the Pigeonhole Principle.

Proof of Pigeonhole Principle

We'd like to prove that if $f: \mathbb{N}_m \to \mathbb{N}_n$ is injective then $m \leq n$. We will do this by induction on n.

Base case $(n=1)$: Consider $f: \mathbb{N}_m \to \mathbb{N}_1$ injective and note that we must then have $f(i)=1 \ \forall i \in \mathbb{N}_m$ but injectivity requires $f(i)=f(j) \Rightarrow i=j$, so there is only one element in $\mathbb{N}_m$, so $m=1$. We have in this case $\underset{1}{\underset{\shortparallel}{m}} \leq \underset{1}{\underset{\shortparallel}{n}}$.

Inductive step: Here we assume that

$$\underbrace{\forall m\,(\text{if there is an injection } f_1 : \mathbb{N}_m \to \mathbb{N}_k, \text{ then } m \leq k)}_{P(k)}$$

and we'd like to prove

$$\underbrace{\forall m\,(\text{if there is an injection } f : \mathbb{N}_m \to \mathbb{N}_{k+1}, \text{ then } m \leq k+1)}_{P(k+1)}.$$

So we start by assuming that $f: \underset{\{1,2,\dots,m\}}{\mathbb{N}_m} \to \underset{\{1,\dots,k+1\}}{\mathbb{N}_{k+1}}$ is an injection.

We consider two cases: (1) $f(i) \leq k \ \forall i \in \mathbb{N}_m$

(2) $\exists i_0 \in \mathbb{N}_m$ s.t. $f(i) = k+1$.

These case are mutually exclusive, and exhaustive, so if we prove it in each case, we're done.

Case 1: If $f(i) \leq k$ $\forall i \in \mathbb{N}_m$, we can define

$f_1 : \mathbb{N}_m \to \mathbb{N}_k$ via $f_1(i) = f(i)$ $\forall i \in m$.

Since f is injective, f_1 is also injective (why)

So by the inductive hypothesis $m \leq k < k+1$

So we are done.

Case 2: Suppose $f(i_0) = k+1$ for some $i_0 \in \mathbb{N}_m$.

Define a new function $g : \mathbb{N}_{m-1} \to \mathbb{N}_m$ by

$$g(i) = \begin{cases} i & \text{when } i < i_0 \\ i+1 & \text{when } i \geq i_0 \end{cases}$$

and notice that g is injective (why?).

Now, let $f_1 = f \circ g : \mathbb{N}_{m-1} \to \mathbb{N}_k$

and notice that f_1 is injective as it is the composition of two injections (proving this is a HW problem)

So, by the inductive hypothesis we have $m-1 \leq k$

and so $m \leq k+1$.

So by induction, we are done. $\blacksquare$

Corollary: Let X & Y be finite, non-empty sets.

If there is an injection $f: X \to Y$
then $|X| \leqslant |Y|$

Proof is an exercise.

Example: If we have a number of pigeons and we place each pigeon in a <u>separate</u> pigeon hole, then

number of pigeons $\leqslant$ number of holes.

Contrapositive: If $|X| > |Y|$ there is <u>no</u> injection $f: X \to Y$.

Thus, if $f: X \to Y$ is a function, $\exists x_1, x_2$, $x_1 \neq x_2$, $f(x_1) = f(x_2)$.

Example: If we have more pigeons than pigeonholes, at least two pigeons must share a pigeonhole.

Example: If you have a drawer with 10 red and 10 blue socks (unpaired), and you are pulling socks from the drawer without looking, what is the minimum number of socks you must draw to guarantee a pair of the same color?

Answer: Three, by the pigeonhole principle. Take $X = \{\text{selected socks}\}$, $Y = \{\text{red}, \text{blue}\}$, then $|Y| = 2$, and we need $|X| \geq 3$ so that we are sure $\exists x_1, x_2$ $f(x_1) = f(x_2)$ while $x_1 \neq x_2$.

Some consequences of the pigeonhole principle:

Def'n: Given a set X, if $|X| = n$ for a non-negative integer n, then we say X is *finite*.

Otherwise, we say X is *infinite*.

Proposition: Suppose $f : X \longrightarrow \mathbb{N}_n$ is an injection. Then X is finite and $|X| \leq n$.

Proof: Exercise.

Proposition: Suppose $X \subseteq Y$, where Y is a finite set. Then X is also finite and $|X| \leqslant |Y|$.

Proof: This is a simple corollary of the previous proposotion. Indeed, let $n = |Y|$ so there is a bijection $f: \mathbb{N}_n \to Y$.
Let $i: X \to Y$ with $i(x) = x$, and note that i is an injection. Thus $f^{-1} \circ i : X \to Y \to \mathbb{N}_n$ is an injection, so by the previous proposition X is finite and $|X| \leqslant n$.

∎

Theorem: Let X, Y be non-empty finite sets with $|X| \leqslant |Y|$ and let $f: X \to Y$ be a function. Then f is <u>not surjective</u>.

Theorem: Let X & Y be non-empty finite sets with $|X| = |Y|$. Then f is an injection if and only if it is surjective.

The proofs are exercises left to the students.

Back to Chapter 10

Counting Principles

Theorem (addition principle)
Suppose X & Y are finite sets with $X \cap Y = \phi$. Then $X \cup Y$ is finite and $|X \cup Y| = |X| + |Y|$.

Proof: We will show that $|X| + |Y| = |X \cup Y|$ which implies $X \cup Y$ is finite.
Let $|X| = n$, $|Y| = m$. If $X = \phi$, then $n = 0$ and $X \cup Y = Y$, and so $|X \cup Y| = |Y| = m + n$.
Similarly, the conclusion holds if $m = 0$.
Let's consider now the case $n \neq 0$ & $m \neq 0$.

Note that there are bijections $f: \mathbb{N}_n \to X$
$g: \mathbb{N}_m \to Y$
and define $h: \mathbb{N}_{n+m} \to X \cup Y$ via

$$h(i) = \begin{cases} f(i) & 1 \leq i \leq n \\ g(i-n) & n+1 \leq i \leq m+n \end{cases}.$$

and note that h is bijective (why?).
So $|X \cup Y| = m + n$.

$\blacksquare$

Corollary: If $X_1, \dots, X_n$ are pairwise disjoint finite sets then

$$|X_1 \cup X_2 \cup \dots \cup X_n| = \sum_{i=1}^{n} |X_i|.$$

Proof: By induction (exercise).

Theorem: (The multiplication principle), Let X, Y be fintie sets. Suppose $|X| = n$, $|Y| = m$, then $|X \times Y| = mn$.

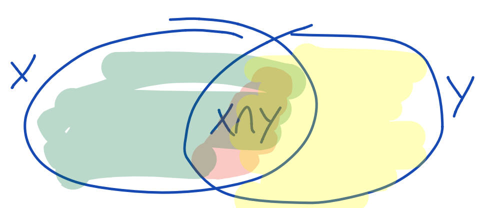
Proof: exercise.

The inclusion-exclusion principle

(I) The case of two sets X and Y.

Theorem: Let X & Y be finite sets, then

$$\boxed{|X \cup Y| = |X| + |Y| - |X \cap Y|.}$$



Proof: Recall that $X \cup Y = (X \setminus Y) \cup (X \cap Y) \cup (Y \setminus X)$ and that $X \setminus Y$, $Y \setminus X$, $X \cap Y$ are pairwise disjoint. So by the addition principle

$$|X \cup Y| = |X \setminus Y| + |Y \setminus X| + |X \cap Y|. \quad \text{— (1)}$$

However $X = (X \setminus Y) \cup (X \cap Y)$ so

$$|X| = |X \setminus Y| + |X \cap Y| \quad \text{— (2)}$$

and similarly $|Y| = |Y \setminus X| + |X \cap Y|$. — (3)

Combining (1), (2), (3), we obtain

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

(II) General inclusion-exclusion principle.

Theorem: Let $A_1, \ldots, A_n$ be finite sets, and for $I = \{i_1, i_2, \ldots, i_r\} \subseteq \mathbb{N}_n$, define

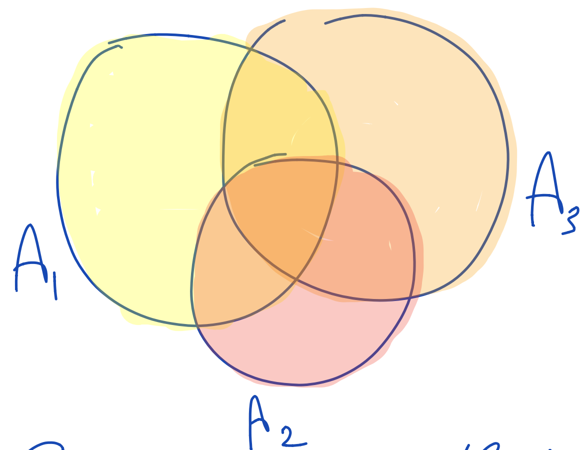
$$A_I = \bigcap_{i \in I} A_i = A_{i_1} \cap A_{i_2} \cdots \cap A_{i_r},$$

then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{I \subseteq \mathbb{N}_n,\, I \neq \emptyset} (-1)^{|I-1|} |A_I|.$$

So for example,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2|$$
$$- |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$



Example: Given 144 tiles that are either triangular or square, red & or blue, wooden or plastic. Suppose there are

68 wooden tiles
69 red tiles
75 triangular tiles

and 36 red wooden tiles
40 triangular wooden tiles
38 red triangular tiles

and 23 red wooden triangular tiles.

How many blue plastic square tiles?

Solution: Let $A_1 = \{\text{blue tiles}\}$
$A_2 = \{\text{Plastic tiles}\}$
$A_3 = \{\text{square tiles}\}$

We want $|A_1 \cap A_2 \cap A_3|$, and we'll get it by the inclusion/exclusion principle, but first we observe that $(A_1 \cap A_2 \cap A_3)^c = A_1^c \cup A_2^c \cup A_3^c$,

so now, we want

$$|A_1^c \cup A_2^c \cup A_3^c|.$$

$$|A_1^c \cup A_2^c \cup A_3^c| = \underbrace{|A_1^c|}_{69} + \underbrace{|A_2^c|}_{68} + \underbrace{|A_3^c|}_{75} - \underbrace{|A_1^c \cap A_2^c|}_{36} - \underbrace{|A_1^c \cap A_3^c|}_{38}$$

$$- \underbrace{|A_2^c \cap A_3^c|}_{40} + \underbrace{|A_1^c \cap A_2^c \cap A_3^c|}_{23}$$

$$= 121.$$

So we have 121 blue plastic square tiles.